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AN INTERFACE CRACK BETWEEN ELASTIC MATERIALS WHEN THERE IS DRY FRICTION†

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An analytic solution of the plane problem of a crack (finite or semi-infinite) along the interface between two elastic half-planes is given. Under tensile and shear forces, the crack opens over an interval (unknown in advance). In the vicinity of the crack tips the edges join smoothly and Coulomb's law of dry friction applies. The materials are perfectly bonded everywhere except along the crack. A closed exact solution is found in the case of a semi-infinite crack. The slip direction, the slip zone length, and formulae for the contact stress and displacement jumps are determined. The problem of a finite crack is reduced to the vector (third-order) Riemann problem in the theory of analytic functions, for which an effective solution is constructed by the method proposed in [1]. An explicit relationship between the smaller and larger slip zone lengths is found by asymptotic analysis. A numerical analysis is carried out. Situations are determined in which the coefficient of friction has practically no effect on the length of the slip zone (to within 5%) and when the effect is substantial (20% or more). An effective analytic solution is found for Comninou's equation [2], which corresponds to the problem of an interface crack ignoring the friction between its edges.

The introduction of contact slip zones without friction in order to prevent the crack edges from going too far in the direct passage from bonding to separation [3] was proposed in [2, 4], where the problem was reduced to a singular integral equation, which was solved numerically. The exact solution of this equation was constructed in [5, 6]. The explicit solution of the problem of a crack with one section along which the edges overlap (ignoring friction) in a uniform stress field was found in [7] (this solution was used there to find an asymptotic solution of the problem with two sections along which the edges overlap) and in [8]. The case of a non-uniform stress field was considered in [9]. In [10] it was proposed take dry friction into account in the contact slip zone. In [11] the problem of an interface crack with friction was reduced to a singular integral equation, for which a numerical method was used. The problem of a semi-infinite interface crack was considered in [12] using other models.‡

1. A SEMI-INFINITE INTERFACE CRACK WHEN THERE IS CONTACT FRICTION

Consider an elastic plane consisting of two half-planes Π_1 and Π_2 with constants of elasticity G_1, ν_1 ($\Pi_1: y > 0$) and G_2, ν_2 ($\Pi_2: y < 0$). Along the interface between the media there is a semi-infinite crack ($0 < x < \infty, y = \pm 0$) which is acted on by concentrated normal and tangential loads (Fig. 1)

$$\sigma_y\Big|_{y=\pm 0} = -P\delta(x-b), \;\; \tau_{xy}\Big|_{y=\pm 0} = -T\delta(x-b), \;\; a < x < \infty \tag{1.1}$$

applied to the crack edges at a given point $x = b$. The crack is open over the interval ($a < x < \infty$): the tangential and normal displacements u and υ undergo a jump with

$$\langle \upsilon \rangle(x) \equiv \upsilon(x,-0) - \upsilon(x,+0) \le 0, \;\; a < x < \infty \tag{1.2}$$

At an a priori unknown point a ($0 < a < b$) the crack edges join smoothly

$$\langle \upsilon \rangle(x) = 0, \;\; 0 < x < a; \;\; \frac{\partial \upsilon}{\partial x}(x, \pm 0) \to 0, \;\; x \to a \pm 0 \tag{1.3}$$

Since the normal displacements are continuous in the slip interval, the stresses σ_y (x, ± 0) must be compressive [2, 13].

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‡See also KIPNIS L. A., The slip line at the tip of an interface crack between different media. Uman, 1989. Deposited at Ukrain. Nauchno-Issl. Inst. Nauchno-Tekhn. Inform. 2.01.89. No. 78.

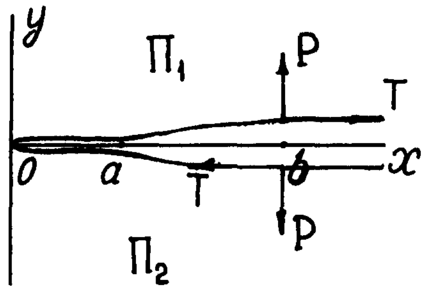

Fig. 1.

$$\sigma_y(x,\pm 0) \leqslant 0, \quad 0 < x < a$$

In the interval $0 < x < a$ the tangential displacement is discontinuous and the tangential and normal stresses are related by the law of dry friction

$$\tau_{xy} = \mu\sigma_y, \quad y = \pm 0, \quad 0 < x < a \tag{1.4}$$

where $|\mu|$ is the coefficient of friction. The sign of μ is verified a posteriori from the condition $\mu = \operatorname{sgn}\langle u\rangle(x)$ for $0 < x < a$. For example, for the chosen parameters of the problem $\mu < 0$ will be considered correct if it turns out that $u(x, +0) > u(x, -0)$ everywhere inside the slip zone (in this case $\tau_{xy}(x, \pm 0) > 0$ by (2.4)) for $0 < x < a$.

We introduce the jumps

$$\chi_1(x) = \langle \partial \upsilon / \partial x\rangle, \quad \chi_2(x) = \langle \partial u / \partial x\rangle, \quad |x| < \infty \tag{1.5}$$

$$\operatorname{supp}\chi_1 \subset [a,\infty), \quad \operatorname{supp}\chi_2 \subset [0, \ \infty)$$

and use them to express the stresses σ_y and τ_{xy} for $y = \pm 0$

$$\begin{gathered}
\sigma_y(x,\pm 0) = -\frac{\kappa\mu_+}{\pi}\int_a^\infty \frac{\chi_1(\xi)}{\xi - x}d\xi + \kappa\mu_-\chi_2(x) \\
\tau_{xy}(x,\pm 0) = -\frac{\kappa\mu_+}{\pi}\int_0^\infty \frac{\chi_2(\xi)}{\xi - x}d\xi - \kappa\mu_-\chi_1(x), \quad |x| < \infty \\
\kappa = \frac{2G_1}{(\sigma + \kappa_1)(1 + \sigma\kappa_2)}, \quad \mu_\pm = \pm\sigma\kappa_2^\pm + \kappa_1^\pm, \quad \kappa_j^\pm = \frac{\kappa_j \pm 1}{2} \\
\kappa_j = 3 - 4\nu_j, \quad j = 1,2; \quad \sigma = G_1G_2^{-1}
\end{gathered} \tag{1.6}$$

Substituting (1.6) into (1.1) taking (1.4) into account we obtain the system

$$\begin{gathered}
-\frac{1}{\pi}\int_a^\infty \frac{\chi_1(\xi)}{\xi - x}d\xi + \gamma\chi_2(x) = -\frac{P}{\kappa\mu_+}\delta(x - b), \quad a < x < \infty \\
-\frac{\mu}{\pi}\int_a^\infty \frac{\chi_1(\xi)}{\xi - x}d\xi + \frac{1}{\pi}\int_0^\infty \frac{\chi_2(\xi)}{\xi - x}d\xi + \gamma\chi_1(x) + \mu\gamma\chi_2(x) = \\
= (\kappa\mu_+)^{-1}(T - \mu P)\delta(x - b), \quad 0 < x < \infty; \quad \gamma = \mu_-\mu_+^{-1}
\end{gathered} \tag{1.7}$$

of two singular integral equations. Using Mellin's theorem on convolution, we can reduce this system to a system of functional equations, which is equivalent to the Riemann problem for a pair of functions

$$\begin{gathered}
\Phi^-(s) = -\frac{\operatorname{ctg}^2\pi s + \gamma^2}{\mu\gamma + \operatorname{ctg}\pi s}\Phi^+(s) + \frac{P_0\operatorname{ctg}\pi s + T_0\gamma}{\mu\gamma + \operatorname{ctg}\pi s}\lambda^{s+1}, \quad s \in \Gamma \\
\Gamma: \operatorname{Re}(s) = \gamma_0 \in (-\varepsilon, 0) \ (0 < \varepsilon < 1), \ \lambda = ba^{-1} > 1
\end{gathered} \tag{1.8}$$

$$\Phi^-(s)=\frac{1}{\kappa\mu_+}\int_0^1\sigma_y(at,0)t^s dt,\ \Phi^+(s)=\int_1^\infty\chi(at)t^s dt$$

$$P_0=(b\kappa\mu_+)^{-1}P,\ T_0=(b\kappa\mu_+)^{-1}T \tag{1.9}$$

We transform the boundary condition (1.8) into

$$\frac{\Phi^-(s)}{L^-(s)}-\Psi^-_{(s)}=L^+(s)\Phi^+(s)+\frac{(P_0\operatorname{ctg}\pi s+T_0\gamma)\lambda^{s+1}}{(\mu\gamma+\operatorname{ctg}\pi s)L^-(s)}-\Psi^-(s) \tag{1.10}$$

$$L^+(s)=\frac{\mu_0\Gamma(-s)\Gamma(1-\alpha-s)}{\Gamma(3/2-s-i\delta)\Gamma(1/2-s+i\delta)},\quad L^-(s)=\frac{\Gamma(1+s)\Gamma(\alpha+s)}{\Gamma(1/2+s+i\delta)\Gamma(1/2+s-i\delta)}$$

$$\alpha=\frac{1}{\pi}\operatorname{arcctg}\mu\gamma\in(0,1),\quad \delta=\frac{1}{2\pi}\ln\frac{1+\gamma}{1-\gamma}$$

$$\mu_0=\frac{\sin\pi\alpha}{\operatorname{ch}^2\pi\delta},\quad \Psi^-(s)=\sum_{n=1}^\infty\frac{C_n}{s-s_n}$$

$$s_{2n-1}=1/2-i\delta-n,\quad s_{2n}=\overline{s_{2n-1}},\quad C_{2n}=\overline{C_{2n-1}}\ (n=1,2,\ldots) \tag{1.11}$$

$$C_{2n-1}=(P_0-iT_0\gamma\operatorname{ctg}\pi\delta)\frac{\sin\pi\alpha\Gamma(n-\alpha+1/2+i\delta)\Gamma(n+i\delta-1/2)}{2\pi\Gamma(n)\Gamma(n+2i\delta)\lambda^{n+i\delta-3/2}} \tag{1.12}$$

The function $\Psi^-(s)$ is analytic in the domain $D^-(\operatorname{Re}(s)>\gamma_0)$, and in D^+ it has poles at the same points as the second term on the right-hand side in (1.10). The choice of the coefficients C_n in (1.12) enables us to neutralize these poles (the points s_n become removable). Liouville's theorem applied to (1.10) yields the following formulae for the solution of (1.8)

$$\Phi^-(s)=L^-(s)\Psi^-(s),\quad \Phi^+(s)=\frac{1}{L^+(s)}\left[\Psi^-(s)-\frac{(P_0\operatorname{ctg}\pi s+T_0\gamma)\lambda^{s+1}}{(\operatorname{ctg}\pi s+\mu\gamma)L^-(s)}\right] \tag{1.13}$$

Taking into account that $L^\pm(s)=O(s^{\mp\alpha})$, $s\to\infty$, $s\in D^\pm$ and comparing (1.9) with (1.13), we find by Tauberian-type theorems that $\sigma_y(x,0)$ and $\chi_1(x)$ have integrable singularities as $x\to a-0$ and $x\to a+0$, respectively. To make sure that the crack edges join smoothly at $x=a$, which is the second condition in (1.3), it is necessary and sufficient that

$$\sum_{n=1}^\infty C_n=0 \tag{1.14}$$

(in this case $\Psi^-(s)=O(s^{-2})$, $s\to\infty$, $|s-s_n|>\varepsilon$, ε being a positive quantity as small as desired; $n=1,2,\ldots$). Substituting (1.12) into (1.14) we can write (1.14) in the form

$$\begin{aligned}&\operatorname{Re}\{\Gamma(3/2+i\delta-\alpha)[\Gamma(1+i\delta)]^{-1}(4\lambda)^{-i\delta}(P-iT\gamma\operatorname{cth}\pi\delta)\times\\&\times F(3/2+i\delta-\alpha,\ 1/2+i\delta,\ 1+2i\delta;\ 1/\lambda)\}=0\end{aligned} \tag{1.15}$$

This equality is a transcendental equation for determining $\lambda=b/a$. When $\lambda\to 1+0$, Eq. (1.15) can be represented as follows [4]:

$$\begin{aligned}&\operatorname{Re}\{(P-iT\gamma\operatorname{cth}\pi\delta)\lambda^{-i\delta}[\Gamma(3/2+i\delta-\alpha)\Gamma(\alpha-1)\{\Gamma(-1/2+i\delta+\alpha)\}^{-1}\times\\&\times F(3/2+i\delta-\alpha,\ 1/2+i\delta,\ 2-\alpha;\ 1-1/\lambda)+(1-1/\lambda)^{\alpha-1}\Gamma(1-\alpha)\times\\&\times F(-1/2+i\delta+\alpha,\ 1/2+i\delta,\alpha;\ 1-1/\lambda)]\}=0\end{aligned}$$

Note that when $G_1\to\infty$, this equation is not the same as the corresponding equation [1] for a semi-infinite punch. This is because in the punch problem the length of the slip zone is determined from the boundedness of the solution at the point of transition from slippage to bonding, while the length of the corresponding zone in the crack problem is found from the boundedness when passing from slippage to separation.

If $\mu = 0$ ($\alpha = 1/2$), Eq. (1.15) has an explicit solution, which is identical with that obtained before in [15]

$$\lambda_k = \mathrm{ch}^2 \frac{\pi}{2\delta}\left(k + \frac{1}{2} - \theta\right), \quad \theta = \frac{1}{\pi}\mathrm{arctg}\left(\frac{T\gamma}{P}\mathrm{cth}\,\pi\delta\right), \quad k = 0, 1, \ldots$$

(for $P > 0$ and $T \to \pm\infty$ we have $\theta \to \pm 1/2$, and for $T = 0$ we have $\theta = 0$).

When $\mu \neq 0$, there is also a denumerable set of roots

$$\sigma_k^{(m)} = \frac{1}{\pi}\mathrm{arctg}\left.\frac{-A(\sigma)}{B(\sigma)}\right|_{\sigma=\sigma_k^{(m-1)}} + k \quad (m = 1, 2, \ldots), \quad \sigma_k^{(0)} = k$$

$$A(\sigma) + iB(\sigma) = (P - iT\gamma\,\mathrm{cth}\,\pi\delta)\Gamma(3/2 + i\delta - \alpha)[\Gamma(1 + i\delta)]^{-1} \times$$

$$\times F(3/2 + i\delta - \alpha,\ 1/2 + i\delta;\ 1 + 2i\delta;\ 4e^{-\pi\sigma/\delta})$$

$$\mathrm{Im}\{A(\sigma), B(\sigma)\} = 0, \quad \lambda_k = 1/4\, e^{\pi\sigma_k/\delta}$$

which is an effective iterative scheme for $k \geqslant 1$.

Below we present a number of initial roots λ_k for $T/P = 10$, $\sigma = 0.01$, $\nu_1 = \nu_2 = 0.3$

$\mu = 0$:	1.326	5.846×10^{14}	4.604×10^{29}	3.626×10^{44}
$\mu = -0.5$:	1.283	5.424×10^{14}	4.272×10^{29}	3.365×10^{44}

In order to determine the root λ_k and the sign of μ corresponding to a physical solution, we obtain computational formulae for the displacement jumps and contact stresses. We set $\chi_1^0(x) = \langle v\rangle(x)$ and $\chi_2^0(x) = \langle u\rangle(x)$. Then, by (1.9) and (1.13), we find using the inverse Mellin transform that

$$\chi_1^0(x) = -\frac{a}{2\pi i}\int_\Gamma \frac{\Phi^+(s)}{s}\left(\frac{x}{a}\right)^{-s} ds + C_1^0$$

$$\chi_2^0(x) = -\frac{a}{2\pi i}\left[\Phi^-(s) + \mathrm{ctg}\,\pi s\Phi^+(s) - P_0\lambda^{s+1}\right]\frac{(x/a)^{-s}}{\gamma s}ds$$

where C_1^0 is a constant fixed by the condition $\chi_1^0(a) = 0$. Using the Cauchy theorem, we compute the last two integrals

$$\chi_1^0(x) = \frac{a}{\pi\mu_0 sh\pi\delta}\mathrm{Re}\left\{i\cos\pi(\alpha + i\delta)\left[\left(\frac{x}{a}\right)^{1/2 - i\delta}\Lambda_0\left(\frac{x}{a}\right) - \Lambda_0(1)\right]\right\} +$$

$$+ b\left[\Pi_0(P_0, T_0;\ x/b) - \Pi_0(P_0, T_0;\ 1/\lambda)\right], \quad a < x < \infty$$

$$\chi_2^0(x) = \frac{a\gamma}{\pi\mu_0}(\mu^2 + 1)\sin^2\pi\alpha\Delta_0\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)^{\alpha}, \quad 0 < x < a$$

$$\chi_2^0(x) = \frac{a\sin\pi\alpha}{\pi\mu_0}\mathrm{Re}\left[(\mu - i\gamma\,\mathrm{cth}\,\pi\delta)\left(\frac{x}{a}\right)^{1/2 - i\delta}\Lambda_0\left(\frac{x}{a}\right)\right\} +$$

$$+ b\Pi_0(T_0, -P_0;\ x/b), \quad a < x < \infty$$

$$\Lambda_k(t) = \sum_{n=1}^{\infty}\frac{\Gamma(n - 1/2 + k + i\delta)\Gamma(n - 1/2 + i\delta + \alpha)}{\Gamma(n)\Gamma(n + 2i\delta)t^n}\Psi^-\left(n - \frac{1}{2} + i\delta\right)$$

$$\Delta_k(t) = \sum_{n=1}^{\infty}\frac{|\Gamma(n + \alpha - 1/2 + i\delta)|^2}{\Gamma(n)\Gamma(n + \alpha - k)}\Psi^-(1 - n - \alpha)t^{n-1} \quad (k = 0, 1)$$

$$\Pi_0(P,T;t)=\frac{\mathrm{ch}^2\pi\delta}{\pi}\mathrm{Re}\left\{(P+iT\gamma\,\mathrm{cth}\,\pi\delta\,\mathrm{sgn}\ln t)\sum_{n=1}^{\infty}\frac{e^{(1/2-i\delta-n)|\ln t|}}{1/2-i\delta-n}\right\}$$

Now we shall find the contact stresses. To begin with, let $0< x < a$. Then

$$\sigma_y(x,0)=\frac{\kappa\mu_+}{2\pi i}\int_\Gamma\Phi^-(s)\left(\frac{x}{a}\right)^{-s-1}ds=\frac{\kappa\mu_+}{\pi\sin\pi\alpha}\left[-\mathrm{ch}^2\pi\delta\Omega\left(\frac{x}{a}\right)+\right.$$

$$\left.+|\cos\pi(\alpha+i\delta)|^2(x/a)^{\alpha-1}\Delta_1(x/a)\right],\quad \tau_{xy}(x,0)=\mu\sigma_y(x,0) \tag{1.16}$$

$$\Omega(t)=\sum_{n=1}^{\infty}\frac{|\Gamma(n+1/2+i\delta)|^2}{\Gamma(n)\Gamma(n+1-\alpha)}\Psi^-(-n)t^{n-1}$$

For $-\infty < x < 0$, by (1.6) we have

$$\sigma_y(x,0)=-\frac{\kappa\mu_+}{\pi}\int_a^\infty\frac{\chi_1(\xi)d\xi}{\xi-x},\quad \tau_{xy}(x,0)=-\frac{\kappa\mu_+}{\pi}\int_0^\infty\frac{\chi_2(\xi)d\xi}{\xi-x}$$

The last two integrals, which are Mellin-type convolutions, can be evaluated using the theorem on convolution in residue theory

$$\sigma_y(x,0)=-\kappa\mu_+(\pi\mu_0)^{-1}\Omega(x/a),\quad -a<x<0$$

$$\tau_{xy}(x,0)=-\frac{\kappa\mu_+}{\pi\mu_0}\left[\mu\Omega\left(\frac{x}{a}\right)+\left(-\frac{x}{a}\right)^{\alpha-1}\gamma\sin\pi\alpha(\mu^2+1)\Delta_1\left(\frac{x}{a}\right)\right],\quad -a<x<0$$

$$\sigma_y(x,0)=\kappa\mu_+\left[\frac{2}{\pi\mu_0 sh2\pi\delta}\mathrm{Re}\left\{i\cos\pi(\alpha+i\delta)\left(-\frac{x}{a}\right)^{-1/2-i\delta}\Lambda_1\left(\frac{x}{a}\right)\right\}-\right.$$

$$\left.-\Pi_1(P_0,T_0;-x/b)\right],\quad -\infty<x<-a$$

$$\tau_{xy}(x,0)=\kappa\mu_+\left[\frac{\sin\pi\alpha}{\pi\mu_0 ch\pi\delta}\mathrm{Re}\left\{(\mu-i\gamma\,\mathrm{cth}\,\pi\delta)\left(-\frac{x}{a}\right)^{-1/2-i\delta}\Lambda_1\left(\frac{x}{a}\right)\right\}-\right.$$

$$\left.-\Pi_1(T_0,-P_0;-x/b)\right],\quad -\infty<x<-a \tag{1.17}$$

$$\Pi_1(P,T;t)=-ch\pi\delta\left[\pi t^{1/2}(1+t)\right]^{-1}(P\cos\delta\ln t+T\gamma\,\mathrm{cth}\,\pi\delta\sin\delta\ln t)$$

As in the case of $\mu = 0$ [15], for $\mu \neq 0$ the jump $\langle u\rangle(x)$ has constant sign in the slip interval only for the root λ_0 of Eq. (1.15). For other roots of this equation $\langle u\rangle(x)$ is a function of variable sign. In the interval $(0, a)$ the normal contact stress corresponding to λ_0 is compressive. We therefore set $\lambda = \lambda_0$ in what follows.

In Fig. 2 curves 1 and 2, respectively, represent graphs of the contact stresses $\sigma_y(x, 0)$ and $\tau_{xy}(x, 0)$ for $\nu_1 = \nu_2 = 0.3$, $\sigma = 0.01$ (in this case $\gamma = 0.2801$), $b = 1$, $P = 0.1$, and $T = 1$ for $\mu = -0.5$ (with $a = 0.7795$). The dashed curves represent the graphs for $\mu = 0$ ($a = 0.7539$). In the same figure and for the same parameter values we present graphs of the normal and tangential displacement jumps $-\chi_1^0(x)G_1$ (A) and $-\chi_2(x)G_1$ (B). It turns out that for $\mu = -0.5$

$$\begin{aligned}&\sigma_y(x,0)<0,\ \ \tau_{xy}(x,0)>0,\ \ \chi_1^0(x)=0,\ \ \chi_2^0(x)<0,\ \ 0<x<a\\&\sigma_y(x,0)=0,\ \ \tau_{xy}(x,0)=0,\ \ \chi_1^0(x)<0,\ \ \chi_2^0(x)<0,\ \ a<x<b\end{aligned} \tag{1.18}$$

The corresponding root λ_0 of Eq. (1.15) for $\mu = 0.5$ gives rise to a solution which satisfies all the inequalities in (1.18) except one, namely, $\tau_{xy}(x, 0) < 0$ for $0 < x < a$ (the jump of the tangential displacement $\chi_2^0(x)$ being negative in the slip zone as before). This case ($\mu = 0.5$) is therefore non-physical. This situation is also encountered for any other relationships between the loads P ($\geqslant 0$) and T when $0 < \gamma \leqslant 1/2$ ($-1/2 \leqslant \gamma \leqslant 1/2$) being the admissible values of $\gamma \neq 0$). This fact is especially undesirable when $P = 0$, $T < 0$. In this case ($\nu_1 = \nu_2 = 0.3$, $\sigma = 0.01$) for $b = 1$ it turns out that $a = 5.47 \times 10^{-15}$ for

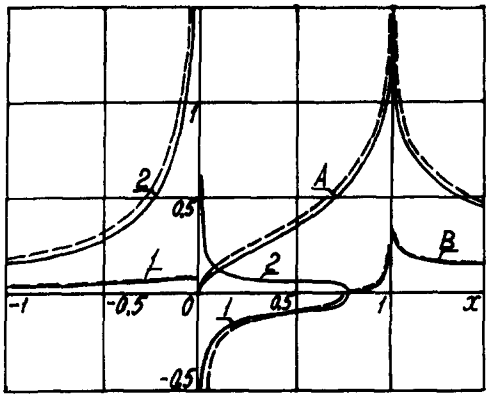

Fig. 2.

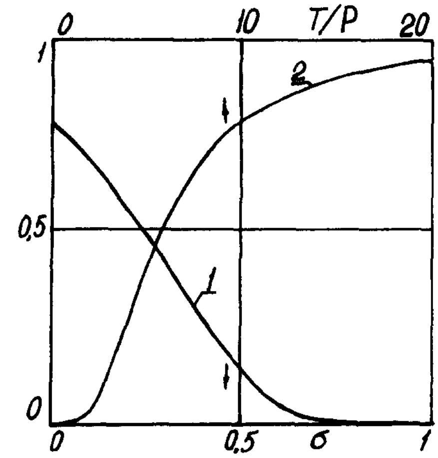

Fig. 3.

$\mu = -0.5$, $a = 5.08 \times 10^{-15}$ for $\mu = 0$, and $a = 4.73 \times 10^{-15}$ for $\mu = 0.5$. The functions $\chi^0{}_2(x)$ and $\sigma_y(x, 0)$ are negative over the whole interval $0 < x < a$ both for $\mu = -0.5$ and $\mu = 0.5$. However, the shear stresses are positive over the slip interval only for $\mu < 0$. Analysis of the case $-1/2 \leqslant \gamma < 0$ leads to the following conclusion: sgn $\mu = -$sgn γ and is independent of the sign of $T(P \geqslant 0)$.

As in [15] for $\mu = 0$, in the case when $\mu \neq 0$ the normal displacement jump changes its sign at some distance from b and oscillates at infinity. For $P = 0$, $T < 0$, and $\gamma > 0$ we have $\chi_1^0(x) < 0$ and $\chi_2^0(x) > 0$ in the interval $a < x < b$, and $\chi_1^0(x)$ changes its sign for the first time at $x_0 = 1 + \varepsilon$ (ε being small and $b = 1$). As T/P increases from $-\infty$ to $+\infty$, the value of x_0 increases from $1 + \varepsilon$ to $A \gg 1$.

As can be seen from the graphs (Fig. 2), the shear and normal stresses increase as $x \to -0$, the shear stress becoming infinite while the normal stress remains bounded. Analysis of (1.16) and (1.17) confirms this fact

$$x \to +0\colon\ \sigma_y(x,0) = O(x^{\alpha-1}),\ \ \tau_{xy}(x,0) = \begin{cases} O(x^{\alpha-1}), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$x \to -0\colon\ \sigma_y(x,0) = O(1),\ \ \tau_{xy}(x,0) = O(x^{\alpha-1})$$

The tangential and normal displacement jumps have a logarithmic singularity at the point $x = b$ ($b = 1$) where the load is applied.

We shall analyse the dependence of the length $a = b/\lambda$ of the slip zone on the coefficient of friction $|\mu|$. In Table 1 we present the values of λ for $T = P = 1$ for some values of γ and μ.

For $\gamma = 0.1$ it can be seen that λ decreases only by 2.5% as $|\mu|$ increases from 0 to 0.5 and by 5.8% as $|\mu|$ increases from 0 to 1.0. The larger $|\gamma|$, the more important it becomes to take friction into account in the slip zone: for $\gamma = 0.5$, λ decreases by 12.3% (by 22.8%) as $|\mu|$ varies from 0 to 0.5 (to 1). It turns out that the smaller the ratio T/P the more significant (expressed as a percentage) is the dependence of the slip zone length on the coefficient of friction. Below we give the values of λ for $\gamma = 0.5$

T/P	10	5	3	2	1	0	−1
$\mu = 0$:	1.08	1.35	2.11	4.06	22.8	1.99×10^3	1.78×10^5
$\mu = -1$:	1.04	1.21	1.76	3.24	17.6	1.53×10^3	1.36×10^5

For $T/P = 10$, λ decreases by only 3.7% as μ varies from 0 to −1, while for $T/P = -1$ it decreases by 23.6%.

For large $|\mu|$ we have the following values of the slip zone length ($T/P = 1$, $\gamma = 0.5$)

$-\mu$	1	2	3	5	7	10
λ	17.6	14.1	12.1	9.98	9.00	8.24

In Fig. 3 curve 1 represents the dependence of $1/\lambda$ on $\sigma = G_1/G_2$ for $\nu_1 = \nu_2 = 0.3$, $T/P = 10$ and $\mu = -0.5$, and curve 2 represents the dependence of the same function on T/P for $\nu_1 = \nu_2 = 0.3$, $\sigma = 0.01$ and $\mu = -0.5$.

Table 1

$-\mu$	$\gamma = 0.1$	0.2	0.3	0.4	0.5
	$10^{-8}\lambda$	$10^{-2}\lambda$	λ	λ	λ
0	120	483	725	85.1	22.8
0.1	119	478	714	83.3	22.3
0.3	118	468	691	79.9	21.1
0.5	117	458	669	76.5	20.0
1.0	113	433	616	68.7	17.6

2. THE EXTENSION WITH SHEAR OF A FINITE INTERFACE CRACK WHEN THERE ARE REGIONS OF DRY FRICTION

Suppose the combined plane $\Pi_1 \cup \Pi_2$ is loaded at infinity

$$\sigma_y = \sigma_0, \ \tau_{xy} = -\tau_0, \ |x| < \infty$$
$$\sigma_x = (1 - 2\gamma \operatorname{sgn} y)\sigma_0, \ \tau_{xy} = -\tau_0, \ |y| < \infty$$

(γ is the parameter defined in (1.7)), suppose a crack ($0 < x < a, y = \pm 0$) is open over an interval ($b_1 < x < b_2$), and for $0 < x < b_1$ and $b_2 < x < a$ suppose the crack edges are in contact and obey the law of dry friction. Over the intervals $-\infty < x < 0$ and $a < x < \infty$ the half-planes are completely banded. Without loss of generality, we can assume that $\tau_0 > 0$ and $\gamma > 0$. Then the left slip zone is smaller, i.e. $b_1 \ll b_2$ (Fig. 4).

The solution of the problem can be represented as the sum of two terms. The first (the elementary solution) corresponds to the case when there is no crack and has the form

$$\sigma_y = \sigma_0, \ \tau_{xy} = -\tau_0, \ \sigma_x = (1 - 2\gamma \operatorname{sgn} y)\sigma_0$$
$$\langle u \rangle = \langle v \rangle = 0, \ |x| < \infty$$

The other term is a solution of the following problem H

$$\sigma_y\big|_{y=\pm 0} = -\sigma_0, \ \tau_{xy}\big|_{y=\pm 0} = \tau_0, \ b_1 < x < b_2$$
$$\langle \sigma_y \rangle = \langle \tau_{xy} \rangle = 0, \ \left\langle \frac{\partial v}{\partial x} \right\rangle = \chi_1(x), \ \left\langle \frac{\partial u}{\partial x} \right\rangle = \chi_2(x), \ |x| < \infty \tag{2.1}$$

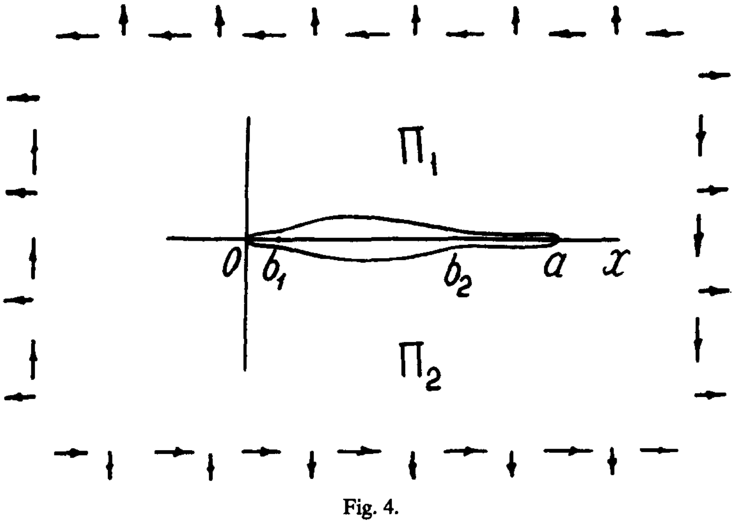

Fig. 4.

$$\operatorname{supp}\chi_1 \subset (b_1, b_2),\ \operatorname{supp}\chi_2 \subset (0, a)$$

$$(\tau_{xy} + \mu\sigma_y)_{y=\pm 0} = \tau_0 - \mu\sigma_0,\ 0 < x < b_1;\ (\tau_{xy} - \mu\sigma_y)_{y=\pm 0} = \tau_0 + \mu\sigma_0,\ b_2 < x < a \tag{2.2}$$

The coefficient of friction μ (by the assumption $\gamma > 0$ and the analysis in Section 1 μ is positive) takes different signs because the slip direction in $(0, b_1)$ is opposite to that in (b_2, a). Furthermore, additional conditions expressed by the inequalities

$$\langle v \rangle(x) \leq 0,\ b_1 < x < b_2$$

$$\sigma_0 + \sigma_y(x, \pm 0) \leq 0,\ x \in (0, b_1) \cup (b_2, a) \tag{2.3}$$

must be satisfied. The latter inequality means that the normal stress in the original problem is compressive in the region where edge joining occurs. The points b_1 and b_2 are to be determined from the smoothness of the crack profile in the vicinity of these points. The cut closure conditions

$$\int_{b_1}^{b_2} \chi_1(x)dx = 0,\ \int_0^a \chi_2(x)dx = 0 \tag{2.4}$$

complete the formulation of the problem.

We shall reduce the above problem H to a system of two singular integral equations. To do this, taking (1.5) and (2.1) into account, we substitute the representations (1.6) of contact stresses in terms of $\chi_1(x)$, $\chi_2(x)$ into the boundary condition (2.2) and obtain

$$\frac{\mu\mu_+}{\pi}\int_{b_1}^{b_2} \frac{\chi_1(\xi)}{\xi - x}d\xi + \mu_-\chi_1(x) + \frac{\mu_+}{\pi}\int_0^a \frac{\chi_2(\xi)}{\xi - x}d\xi - \mu\mu_-\chi_2(x) = \tau_+,\ 0 < x < b_2$$

$$-\frac{\mu\mu_+}{\pi}\int_{b_1}^{b_2} \frac{\chi_1(\xi)}{\xi - x}d\xi + \mu_-\chi_1(x) + \frac{\mu_+}{\pi}\int_0^a \frac{\chi_2(\xi)}{\xi - x}d\xi + \mu\mu_-\chi_2(x) = \tau_-,\ b_1 < x < a, \tag{2.5}$$

$$\tau_\pm = \kappa^{-1}(-\tau_0 \pm \mu\sigma_0)$$

where $\mu_\pm$ and κ are defined in (1.6).

We will seek a solution of (2.5) in the class of Hölder continuous functions such that $\chi_1(x)$ is bounded at $x = b_1$ and $x = b_2$ and $\chi_2(x)$ has integrable singularities at $x = 0$ and $x = a$ is bounded as $x \to b_1$ and $x \to b_2$. We set

$$\lambda_j = b_j / a\ (j = 1, 2),\ t_\pm(x) = -\kappa^{-1}(\tau_{xy} \pm \mu\sigma_y)(x, 0)$$

and introduce the piecewise-analytic functions

$$\Phi_1^-(s) = \int_{\lambda_1/\lambda_2}^{1} \chi_1(b_2\tau)\tau^s d\tau,\ \Phi_1^+(s) = \int_1^{\lambda_2/\lambda_1} \chi_1(b_1\tau)\tau^s d\tau$$

$$\Phi_2^-(s) = \int_0^1 \chi_2(a\tau)\tau^s d\tau,\ \Phi_2^+(s) = \int_1^\infty t_+(b_2\tau)\tau^s d\tau \tag{2.6}$$

$$\Phi_3^-(s) = \int_0^1 t_-(b_1\tau)\tau^s d\tau,\ \Phi_3^+(s) = \int_1^\infty t_-(a\tau)\tau^s d\tau$$

with jump line Γ defined in (1.8). Using the Mellin transform we reduce (2.5) to the Riemann vector problem

$$\Phi_1^+(s) = (\lambda_1/\lambda_2)^{-s-1}\Phi_1^-(s)$$

$$\Phi_2^+(s) = l_{11}(s)\Phi_1^-(s) + \lambda_2^{-s-1} l_{12}(s)\Phi_2^-(s) - \tau_+(s+1)^{-1}$$

$$\Phi_3^+(s) = \lambda_2^{s+1} l_{21}(s)\Phi_1^-(s) + \lambda_{22}(s)\Phi_2^-(s) - \lambda_1^{s+1}\Phi_3^-(s) - (1 - \lambda_1^{s+1})(s+1)^{-1}\tau_- \tag{2.7}$$

$$l_{j1}(s) = \mu_- - (-1)^j \mu\mu_+ \operatorname{ctg}\pi s,\ l_{j2}(s) = (-1)^j \mu\mu_- + \mu_+ \operatorname{ctg}\pi s\ (j = 1, 2)$$

A method of solving problems of type (2.7) was proposed in [1]. We will state the final result, omitting the construction of the solution

$$\Phi_1^-(s)=\frac{R_1(s)}{L_1^-(s)}+\left(\frac{\lambda_1}{\lambda_2}\right)^{s+1}\frac{\Psi_2^-(s)}{L_2^+(s)},\quad \Phi_1^+(s)=\left(\frac{\lambda_1}{\lambda_2}\right)^{-s-1}\Phi_1^-(s)$$

$$\Phi_2^-(s)=\frac{R_0(s)}{L_0^-(s)}-\frac{\lambda_1^{s+1}l_{11}(s)}{l_{12}(s)L_2^+(s)}\Psi_2^-(s)-\frac{\lambda_2^{s+1}l_{21}(s)}{l_{22}(s)L_1^-(s)}R_1(s)$$

$$\Phi_2^+(s)=-\frac{\tau_+}{s+1}+L_1^+(s)R_1(s)+\frac{\lambda_2^{-s-1}l_{12}(s)}{l_{22}(s)}L_0^+(s)R_0(s)$$

$$\Phi_3^-(s)=(s+1)^{-1}\tau_- - L_2^-(s)\Psi_2^-(s),\quad \Phi_3^+(s)=-(s+1)^{-1}\tau_- + L_0^+(s)R_0(s) \tag{2.8}$$

$$R_0(s)=C+e_0\tau_-(s+1)^{-1}+\Psi_1^+(s),\quad R_1(s)=e_1\tau_*(s+1)^{-1}+\Psi_1^-(s)+\Psi_2^+(s)$$

$$e_0=-\mu_0^{-1}\Gamma(2-\alpha),\quad e_1=-\pi(\delta^2+1/4)[\mu_1\Gamma(2-\alpha)\operatorname{ch}\pi\delta]^{-1},\quad \tau_*=\tau_+-\tau_-$$

$$L_0^+(s)=-\frac{\mu_0\Gamma(-s)}{\Gamma(1-\alpha-s)},\quad L_0^-(s)=\frac{\Gamma(1+s)}{\Gamma(\alpha+s)},\quad \mu_1=\frac{2\mu\mu_+\sin\pi\alpha}{\operatorname{ch}^2\pi\gamma}$$

$$L_1^+(s)=-\frac{\mu_1\Gamma(-s)\Gamma(1-\alpha-s)}{\Gamma(1/2-s-i\delta)\Gamma(1/2-s+i\delta)},\quad L_1^-(s)=\frac{\Gamma(1+s)\Gamma(\alpha+s)}{\Gamma(1/2+s+i\delta)\Gamma(1/2+s-i\delta)}$$

$$L_2^+(s)=-\frac{\mu_1\Gamma(-s)\Gamma(\alpha-s)}{\Gamma(1/2-s-i\delta)\Gamma(1/2-s+i\delta)},\quad L_2^-(s)=\frac{\Gamma(1+s)\Gamma(1-\alpha+s)}{\Gamma(1/2+s+i\delta)\Gamma(1/2+s-i\delta)}$$

$$\Psi_1^+(s)=\sum_{j=1}^{\infty}\frac{A_j^+}{s-j+\alpha},\quad \Psi_1^-(s)=\sum_{j=1}^{\infty}\frac{A_j^-}{s-1+j+\alpha},\quad \Psi_2^\pm(s)=\sum_{j=1}^{\infty}\frac{B_j^\pm}{s\mp s_j}$$

The numbers μ_0 and s_j are defined in (1.11), C is an arbitrary constant, and the coefficients $A_j^\pm$, $B_j^\pm$ will be found later.

By the boundedness of σ_y and τ_{xy} at $x=b_1$ and $x=b_2$, the function $t_-(b_1\tau)$ is bounded as $t\to 1-0$. Taking (2.6) into account, by an Abelian-type theorem, we obtain the asymptotic expression $\Phi_3^-(s)=O(s^{-1})$ as $s\to\infty$ with $s\in D^-$, which holds for the function defined in (2.8) if and only if

$$\sum_{j=1}^{\infty}B_j^-=0 \tag{2.9}$$

By (2.6), the closure conditions (2.4) for the cut have the form $\Phi_1^-(0)=\Phi_2^-(0)=0$, which implies that

$$C+\tau_-e_0+\Psi_1^+(0)=0,\quad \tau_*e_1+\Psi_1^-(0)+\Psi_2^+(0)=0 \tag{2.10}$$

For the functions (2.8) to be analytic in the corresponding half-spaces it is necessary and sufficient that the coefficients

$$A_n^\pm=A_{n1}^\pm+CA_{n0}^\pm,\quad B_n^\pm=B_{n1}^\pm+CB_{n0}^\pm \tag{2.11}$$

should satisfy the following infinite Poincaré–Koch algebraic system

$$\begin{aligned}
A_{nk}^- &= \lambda_2^{n+\alpha-2}p_n^-\left(-\delta_{k0}+\frac{\tau_-e_0}{n+\alpha-2}\delta_{k1}+\sum_{j=1}^{\infty}\frac{A_{jk}^+}{n+j-1}\right)\\
B_{nk}^+ &= \left(\frac{\lambda_1}{\lambda_2}\right)^{s_n+1}q_n^+\sum_{j=1}^{\infty}\frac{B_{jk}^-}{s_n+s_j}\\
A_{nk}^+ &= \lambda_2^{n-\alpha+1}p_n^+\left[\frac{\tau_*e_1}{n-\alpha+1}\delta_{k1}+\sum_{j=1}^{\infty}\left(\frac{A_{jk}^-}{n-1+j}+\frac{B_{jk}^+}{n-\alpha-s_j}\right)\right]\\
B_{nk}^- &= \left(\frac{\lambda_1}{\lambda_2}\right)^{s_n-1}q_n^-\left[\frac{\tau_*e_1}{s_n-1}\delta_{k1}+\sum_{j=1}^{\infty}\left(\frac{A_{jk}^-}{s_n-j+1-\alpha}+\frac{B_{jk}^+}{s_n+s_j}\right)\right]
\end{aligned} \tag{2.12}$$

$(k = 0,1;\ n = 1,2,3\ldots)$

$$p_n^- = \frac{\gamma \operatorname{ch}^2 \pi\delta}{\pi\Gamma^2(n)}\left|\Gamma\left(n+\alpha-\frac{1}{2}+i\delta\right)\right|^2,\quad p_n^+ = -\frac{\gamma d^2(\mu^2+1)}{\pi\Gamma^2(n)}\left|\Gamma\left(n+\frac{1}{2}-\alpha+i\delta\right)\right|^2$$

$$q_n^+ = \mu_1^{-1}s_n q_n,\ q_n^- = \mu_1 s_n^{-1}q_n,\ d = \sin\pi\alpha$$

$$q_{2m} = \frac{i\cos\pi(\alpha+i\delta)\Gamma(m-1/2-i\delta)\Gamma(m+1/2-i\delta)\Gamma(m+\alpha-1/2-i\delta)\Gamma(m+1/2-\alpha-i\delta)}{2\pi \operatorname{sh}\pi\delta\Gamma^2(m)\Gamma^2(m-2i\delta)}$$

$$q_{2m-1} = \overline{q_{2m}}\ (m = 1,2\ldots)$$

Substituting the second relationship from (2.11) into (2.9), we find that

$$C = -\frac{\Lambda_1}{\Lambda_0},\quad \Lambda_k = \sum_{j=1}^{\infty} B_{jk}^-\ (k = 0,1) \tag{2.13}$$

Satisfying conditions (2.10), we arrive at the system

$$\sum_{j=1}^{\infty}\frac{A_j^+}{\alpha-j} - \frac{\Lambda_1}{\Lambda_0} = -\tau_- e_0,\quad \sum_{j=1}^{\infty}\left(\frac{A_j^-}{\alpha-1+j} - \frac{B_j^+}{s_j}\right) = -\tau_* e_1 \tag{2.14}$$

of two transcendental equations in λ_1 and λ_2. Since λ_1 is small, $0 < \lambda_1 < 10^{-4}$, and $\lambda_2 \gg \lambda_1$ [4, 11], it follows that (2.14) can be reduced to a single transcendental equation by approximation. To do this, we first transform the infinite system (2.12) to the form (k = 0, 1)

$$\begin{gathered}
A_{nk}^- = \lambda_2^{n+\alpha-2}p_n^-\left(-\delta_{k0} + \frac{\tau_- e_0}{n+\alpha-2}\delta_{k1} + \sum_{j=1}^{\infty}\frac{A_{jk}^+}{n+j-1}\right) \\
A_{nk}^+ = \lambda_2^{n-\alpha+1}p_n^+\left(\frac{\tau_* e_1}{n-\alpha+1}\delta_{k1} + \sum_{j=1}^{\infty}\frac{A_{jk}^+}{n+j-1}\right)\ (n = 1,2\ldots) \\
B_{nk}^+ = 0\ (n = 1,2,\ldots),\ B_{nk}^- = 0\ (n - 3,4,\ldots) \\
B_{nk}^- = \left(\frac{\lambda_1}{\lambda_2}\right)^{s_n-1} f_{nk},\ f_{nk} = q_n^-\left(\frac{\tau_* e_1}{s_n-1}\delta_{k1} + \sum_{j=1}^{\infty}\frac{A_{jk}^-}{s_n-j+1-\alpha}\right)\ (n = 1,2)
\end{gathered} \tag{2.15}$$

The coefficients $A_{nk}^{\pm}$ can be determined from (2.15) by the recurrence relationships

$$A_{nk}^- = \lambda_2^{n+\alpha-2}\sum_{m=1}^{\infty}a_{nkm}^-\lambda_2^{m-1},\quad A_{nk}^+ = \lambda_2^{n-\alpha+1}\sum_{m=1}^{\infty}a_{nkm}^+\lambda_2^{m-1}$$

$$a_{nk1}^- = p_n^-[-\delta_{k0} + \tau_- e_0(n+\alpha-2)^{-1}\delta_{k1}]$$

$$a_{nk1}^+ = p_n^+[\tau_* e_1(n-\alpha+1)^{-1}\delta_{k1} + \lambda_2^{\alpha-1}n^{-1}a_{1k1}^-]$$

$$a_{nkm}^- = p_n^-\lambda_2^{1-\alpha}\sum_{j=1}^{m-1}\frac{a_{j,k,m-j}^+}{n+j-1},\quad a_{nkm}^+ = p_n^+\lambda_2^{\alpha-1}\sum_{j=1}^{m}\frac{a_{j,k,m+1-j}^-}{n+j-1}$$

$(m = 2,3,\ldots)$

Expression (2.13) for C can be transformed into

$$C = -\frac{\tau_* e_1 + a_1^-}{a_0^-},\quad a_m^+ = \sum_{j=1}^{\infty}\frac{A_{jm}^+}{\alpha-j},\quad a_m^- = \sum_{j=1}^{\infty}\frac{A_{jm}^-}{\alpha+j-1}\ (m = 0,1)$$

and in place of (2.14) we obtain the equation

$$a_0^{-1}(1+a_0^+)(\tau_* e_1 + a_1^-) = \tau_- e_0 + a_1^+$$

for λ_2, the smaller parameter λ_1 being expressed in terms of λ_2 by

$$\lambda_1 = \lambda_2 g^{1/(2i\delta)}, \quad g = -(Cf_{20} + f_{21})(Cf_{10} + f_{11})^{-1} \tag{2.17}$$

A denumerable set of solutions of Eq. (2.16) exists among which it is necessary to choose the maximum value $\lambda_2 \in (0, 1)$ [7], and also the denumerable set of values of λ_1 satisfying the first equality in (2.17) as well as the inequality $\lambda_1 \leqslant \lambda_2$

$$\lambda_1 = \lambda_2 \exp\left\{(2\delta)^{-1} \arg g - \pi l \delta^{-1}\right\} \quad (l = 1, 2, \ldots)$$

$$|\arg g| < \pi$$

However, the physical solution, i.e. one for which inequalities (2.3) are satisfied, corresponds to one and only one value $\lambda_1 = \lambda_2 \exp\{(2\delta)^{-1} \arg g - \pi\delta^{-1}\}$ (the maximum of the values above).

The dependence of the slip zone length on the friction is shown in Fig. 5 which gives graphs of $\lambda^*_2 = 2\lambda_2 - 1$ (to enable a comparison to be made with [11] λ_2^* is taken in place of λ_2) as a function of the coefficient of friction μ in the case when $\sigma_0/\tau_0 = 0$ for $\gamma = 0.1$, $\gamma = 0.3$, and $\gamma = 0.5$ (curves 1–3, respectively). The dashed line represents the graph of λ_2^* for $\gamma = 0.5$ from [11]. As μ varies from 0 to 1, the length $1 - \lambda_2$ of the larger slip zone increases by 26% for $\gamma = 0.5$ and only by 5% for $\gamma = 0.1$. The values of λ_1 and λ_2 for $\mu = 0.3$ and $\sigma_0/\tau_0 = 0$ are presented below for various values of γ

γ	0.1	0.2	0.3	0.4	0.5
λ_1	1.29×10^{-43}	4.94×10^{-20}	9.91×10^{-13}	5.45×10^{-9}	1.15×10^{-6}
λ_2	0.690	0.683	0.673	0.662	0.647

and the values of λ_2 for $\gamma = 0.5$ with $\mu = 0.3$ and $\mu = 10^{-5}$ are presented for some values of σ_0/τ_0 ($\tau_0 > 0$)

σ_0/τ_0	−0.6	−0.4	−0.2	0	0.1	0.2	0.4
$\mu = 0.3$:	0.137	0.239	0.414	0.647	0.754	0.839	0.942
$\mu = 10^{-5}$:	0.175	0.278	0.449	0.671	0.772	0.851	0.950

The last row is consistent with the corresponding results in [16, 7] for $\mu = 0$.

As can be seen, the separation zone increases by a few percent when γ ($0 < \gamma \leqslant 0.5$) increases and μ ($0.5 \geqslant \mu \geqslant 0$) decreases, while it increases several-fold when σ_0/τ_0 varies from −0.6 to 0.6.

The formulae for the jumps of the displacement and the contact stresses can be obtained using the inverse Mellin transform in residue theory. For example, for the normal displacement jump, taking (2.6), (2.8) and (2.15) into account, we find

$$\chi_1^0(x) = I(x) - I(b_1), \quad b_1 < x < b_2$$

$$I(x) = \sum_{m=1}^{\infty} M_m \left[-\frac{b_2}{s_m} B_m^+ \left(\frac{x}{b_2}\right)^{-s_m} + \frac{b_1}{\mu_1} B_m^- \left(\frac{x}{b_1}\right)^{s_m} \right] \tag{2.18}$$

$$M_{2m} = \Gamma(m)\Gamma(m - 2i\delta)[\Gamma(\tfrac{1}{2} + m - i\delta)\Gamma(\alpha - \tfrac{1}{2} + m - i\delta)]^{-1}, \quad M_{2m-1} = \overline{M_{2m}}$$

Because b_1 is small, for $b_1 + \leqslant x < b_2$ ($\varepsilon > 0$) (2.18) can be transformed to the form

$$\chi_1^0(x) = \frac{b_2}{\pi} \operatorname{Re}\left\{ \frac{\cos\pi(\alpha - i\delta)}{i \operatorname{sh} \pi\delta} \sum_{m=1}^{\infty} \frac{\Gamma(m - \tfrac{1}{2} + i\delta)\Gamma(m + \tfrac{1}{2} - \alpha + i\delta)}{\Gamma(m)\Gamma(m + 2i\delta)(\tfrac{1}{2} - m - i\delta)} \times \right.$$

$$\left. \times \left(\frac{\tau_* e_1}{m - \tfrac{3}{2} + i\delta} + \sum_{j=1}^{\infty} \frac{A_j^-}{m + \tfrac{1}{2} + i\delta - \alpha - j} \right) \left(\frac{x}{b_2} \right)^{m - \frac{1}{2} + i\delta} \right\}$$

In Fig. 6 we show the graphs of $-G_1\chi_1^0(x)$ for the case $\sigma = 0.01$, $\nu_1 = 0.1$, $\nu_2 = 0.3$, $\mu = 0.3$ with $\sigma_0/\tau_0 = -0.2, 0, 0.2$ (curves 1^-, 1° and 1^+, respectively). As in [7], the crack opening decreases as the compressive stress intensity increases.

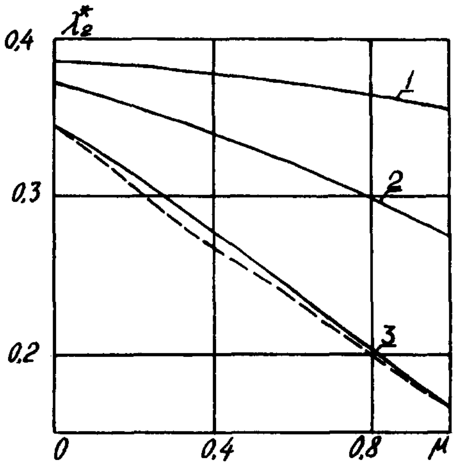

Fig. 5.

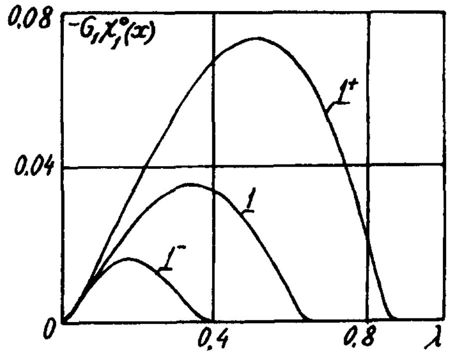

Fig. 6.

3. THE EFFECTIVE SOLUTION OF COMNINOU'S INTEGRAL EQUATION

When $\tau_0 = 0$ and $\mu = 0$, the system of integral equations (2.5) for a crack $\{-a < x < a, y = \pm 0\}$ open over the interval $|x| < b$ can be reduced to the following equation ($0 < \gamma \leqslant 1/2$) [2, 5]

$$2\int_\lambda^1 \frac{\varphi(\eta)\eta}{y^2-\eta^2}\left(1-\frac{\gamma^2\eta}{y}\right)d\eta = \frac{\pi\sigma_0}{\kappa\mu_+},\quad \lambda < y < 1 \tag{3.1}$$

where $\lambda = [1-(b/a)^2]^{1/2}$ and $\varphi(\eta) = \chi_1(a(1-\eta^2)^{1/2}$. The remaining symbols are the same as in Sections 1 and 2. For Eq. (3.1) an exact solution (expressed as a series) was constructed in [5] in terms of elliptic functions. We shall obtain a solution of this equation in a different form lending itself well to numerical realization. We denote by $\varphi_-(x)$, $\varphi_+(x)$ the left-hand side of (3.1) for $0 < x < \lambda$ and $1 < x < \infty$, respectively, and introduce the Mellin transforms

$$\begin{aligned}
&\Phi_1^-(s) = \int_\lambda^1 \varphi(x)x^s dx,\quad \Phi_1^+(s) = \int_1^{1/\lambda} \varphi(\lambda x)x^s dx \\
&\Phi_2^-(s) = \int_0^1 \varphi_-(\lambda x)x^s dx,\quad \Phi_2^+(s) = \int_1^\infty \varphi_+(x)x^s dx
\end{aligned} \tag{3.2}$$

for which we obtain the vector Riemann problem

$$\begin{aligned}
&\Phi_1^+(s) = \lambda^{-s-1}\Phi_1^-(s) \\
&\Phi_2^+(s) = G(s)\Phi_1^-(s) - \lambda^{s+1}\Phi_2^-(s) - f(1-\lambda^{s+1})(s+1)^{-1},\quad s\in\Gamma \\
&\Gamma: \mathrm{Re}(s) = \gamma_0 \in (0,1),\quad f = (\kappa\mu_+)^{-1}\sigma_0 \\
&G(s) = \mathrm{tg}\,\tfrac{1}{2}\pi s + \gamma^2 \mathrm{ctg}\,\tfrac{1}{2}\pi s = K^+(s)K^-(s) \\
&K^+(s) = \frac{(1-\gamma^2)\Gamma(\tfrac{1}{2}-s/2)\Gamma(1-s/2)}{\Gamma(1-s/2-i\beta/2)\Gamma(1-s/2+i\beta/2)},\quad K^-(s) = \frac{\Gamma(s/2)\Gamma(\tfrac{1}{2}+s/2)}{\Gamma(s/2+i\beta/2)\Gamma(s/2-i\beta/2)} \\
&\beta = \pi^{-1}\ln\{(1+\gamma)(1-\gamma)^{-1}\}
\end{aligned} \tag{3.3}$$

Solving (3.3) by the scheme of [17], we find that

$$\begin{aligned}
&\Phi_1^-(s) = [K^-(s)]^{-1}\{\Psi^+(s) + f(s+1)^{-1}[K^+(-1)]^{-1}\} + \lambda^{s+1}\Psi^-(s)[K^+(s)]^{-1} \\
&\Phi_2^-(s) = f(s+1)^{-1} + K^-(s)\Psi^-(s) \\
&\Phi_2^+(s) = -f(s+1)^{-1} + K^+(s)\{\Psi^+(s) + f(s+1)^{-1}[K^+(-1)]^{-1}\}
\end{aligned} \tag{3.4}$$

$$\Psi^+(s)=\sum_{j=1}^{\infty}\frac{A_j^+}{s-s_j},\quad \Psi^-(s)=\sum_{j=1}^{\infty}\frac{A_j^-}{s+s_j-2}$$

$$s_{2j}=2j-i\beta,\quad s_{2j-1}=2j+i\beta\quad (j=1,2,\ldots)$$

The coefficients $A_n^\pm$ form a solution of the infinite Poincaré–Koch system

$$A_m^+=\lambda^{s_m+1}\Delta_m^-\sum_{j=1}^{\infty}\frac{A_j^-}{s_j+s_m-2}$$

$$A_m^-=\lambda^{s_m-3}\Delta_m^+\left(\frac{f_*}{3-s_m}+\sum_{j=1}^{\infty}\frac{A_j^+}{2-s_j-s_m}\right)$$

$$\Delta_{2m-1}^+=+\frac{(1-\gamma^2)}{\pi}\left[\frac{\Gamma(m+i\beta/2)\Gamma(m-\tfrac{1}{2}+i\beta/2)}{\Gamma(m)\Gamma(m+i\beta)}\right]^2,\quad \Delta_{2m}^+=\overline{\Delta_{2m-1}^+}$$

$$\Delta_{2m-1}^-=-\frac{\Delta_{2m-1}^+}{(1-\gamma^2)^2}\left(m-\frac{1}{2}+\frac{i\beta}{2}\right)^2,\quad \Delta_{2m}^-=\overline{\Delta_{2m-1}^-}$$

$$f_*=\tfrac{1}{2}f\pi^{\frac{1}{2}}(\beta^2+1)(1-\gamma^2)^{-1}(\operatorname{ch}\tfrac{1}{2}\pi\beta)^{-1}$$

which can be inverted by means of the recurrence relations

$$A_m^+=\lambda^{s_m}\sum_{k=1}^{\infty}a_{mk}^+\lambda^{2k-2},\quad A_m^-=\lambda^{s_m-3}\sum_{k=1}^{\infty}a_{mk}^-\lambda^{2k-2}$$

$$a_{m1}^-=f_*\Delta_m^+(3-s_m)^{-1}\tag{3.5}$$

$$a_{mn}^+=\sum_{j=1}^{n}\left(\frac{\lambda^{i\beta}a_{2j-1,n+1-j}^-}{s_m+s_{2j-1}-2}+\frac{\lambda^{-i\beta}a_{2j,n+1-j}^-}{s_m+s_{2j}-2}\right)\quad (n=1,2,\ldots)$$

$$a_{mn}^-=\sum_{j=1}^{n-1}\left(\frac{\lambda^{i\beta}a_{2j-1,n-j}^+}{2-s_m-s_{2j-1}}+\frac{\lambda^{-i\beta}a_{2j,n-j}^-}{2-s_m-s_{2j}}\right)\quad (n=2,3,\ldots)$$

The crack edges join one another at $x=\pm b$ if and only if $\Phi_2^-(s)=O(s^{-1})$, $s\to\infty$, $s\in D^-$, which is equivalent to

$$\sum_{j=1}^{\infty}A_j^-=0\tag{3.6}$$

by (3.4). The latter condition is a transcendental equation in λ. Taking into account the fact that λ is small [2, 5] and using (3.5), we transform (3.6) into

$$\lambda^{2i\beta}+(a_{11}^-)^{-1}a_{21}^-+O(\lambda^2)=0,\quad \lambda\to 0$$

Hence we find the explicit approximate values

$$\lambda_k=4\exp\left\{-\frac{1}{2\beta}\operatorname{arctg}\frac{2\beta}{1-\beta^2}-\frac{\pi(k+\tfrac{1}{2})}{\beta}\right\},\quad k=0,1,\ldots$$

for λ. The physical solution corresponds only to the value λ_0 (for $k=0$). For $\gamma=0.4854$ we find $\lambda_0=0.01450$ and $\lambda_*=1-b/a=1-(1-\lambda_0^2)^{1/2}=1.0517\times10^{-4}$, which is identical with the result obtained in [5].

Below we give the values of λ_0 and λ_* for some values of γ

γ	0.1	0.2	0.3	0.4	0.5
λ_0	3.079×10^{-11}	7.664×10^{-6}	5.143×10^{-4}	4.451×10^{-3}	1.712×10^{-2}
λ_*	4.740×10^{-22}	2.937×10^{-11}	1.322×10^{-7}	9.906×10^{-6}	1.465×10^{-4}

By (3.2) and (3.4), the solution of Eq. (3.1) has the form

$$\varphi(x) = -\sum_{j=1}^{\infty}\left[A_j^- E_j\left(\frac{x}{\lambda}\right)^{s_j-3} + G_j\Psi^-(s_j)\left(\frac{x}{\lambda}\right)^{-s_j-1}\right], \quad \lambda < x < 1$$

$$G_{2m} = \frac{\Gamma(m - i\beta/2)\Gamma(m + \frac{1}{2} - i\beta/2)}{\pi(1-\gamma^2)\Gamma(m)\Gamma(m - i\beta)}, \quad G_{2m-1} = \overline{G_{2m}}$$

$$E_{2m} = (m - \tfrac{1}{2} - i\beta/2)\left[\pi(1-\gamma^2)^2 G_{2m}\right]^{-1}, \quad E_{2m-1} = \overline{E_{2m}}$$

For $\tau_0 \neq 0$ the problem can be reduced to two equations, which can be solved independently and differ from (3.1) only by the right-hand sides, i.e. the method presented is applicable in this case.

4. CONCLUSIONS

1. The effect of friction in problems concerned with an interface crack is much less than in problems involving a punch for corresponding parameter values, as the coefficient of friction decreases from 1 to 0.1, the slip zone length in Flamant's problem concerned with a semi-infinite punch when there is dry friction and bonding [1] increases more than 700-fold, while in the corresponding problem of a semi-infinite crack the length of this section decreases by a factor of 1.15.

There is a wide range of real parameter values of the problem in which the stresses and displacement jumps on the interface vary by just a few percent as the coefficient of friction varies from 0 to 0.5 (the shear stresses in the slip domain are an exception). This makes it possible to use an approximate approach when considering the class of contact problems involving an interface crack in the presence of friction.

2. In the problem of the indentation of a punch into a half-plane when there is a bonding zone $(-b, b)$ and slip zones $(-a, -b)$, (b, a), if $\mu \to 0$, then $\lambda = b/a \to 0$, i.e. the bonding domain contracts to a point as the coefficient of friction tends to zero [18, 1]. It follows that Galin's problem becomes the problem of a smooth punch as $\mu \to 0$ (the boundary conditions degenerate), and its solution becomes Sadowsky's solution [19]. On the other hand, for $\mu = 0$ and $\lambda > 0$ we have Fal'kovich's problem [20]. However, in [20] bonding prevails almost everywhere in the contact plane: for $\nu = 0.3$, $b = 0.997a$. This can be explained by the choice in [20] of the root of the corresponding equation from which to determine the accessory parameter $\lambda°$ which appears in the differential equation to which the problem can be reduced: the root λ_2° was taken instead of the value $\lambda_2^\circ = 0$ leading to Sadowsky's solution. For this choice the contact domain consists of one bonded region in the middle and two small slip sections at the end. An incorrect situation arises: the shear stresses in $(0, b)$ have fixed sign, while the normal stresses have variable sign.

The boundary conditions do not degenerate as $\mu \to 0$ in the problem of an interface crack: the separation domain and slip zones remain as before, and the corresponding solution is correct. However, the second (next) root of the corresponding transcendental equation leads to a non-physical solution also, as in the punch problem.

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